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A Note on the Order of Magnitude of Certain Titchmarsh–Weyl m-Functions

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1. INTRODUCTION

In [3] Atkinson considered the problem of deriving upper and lower bounds for the Titchmarsh–Weyl m-functions associated with the linear, second-order, differential equation

$$-(py')' + qy = \lambda wy \qquad 0 \leqslant x < \infty \tag{1.1}$$

in the case where p, q, $w \in L^1_{\text{loc}}(0, \infty)$, $w \geqslant 0$ and λ is a complex parameter with

$$0 < \varepsilon < \arg(\lambda) < \pi - \varepsilon. \tag{1.2}$$

The bounds derived in [3] are superior to those of an earlier publication, [2], both in the sense that they are two sided and because they give sharper estimates in a number of examples.

One of the complications involved in the analysis of (1.1) in the case where there are no restrictions on the sign of p involves the way in which the m-function is defined. A geometric interpretation of the m-function requires us to consider the Weyl disc, $D(X, \lambda)$, which is defined to be the closed interior of the circle which is the image of the real line under the mapping

$$\xi\colon \to \frac{-\{\theta(X, \lambda) - \xi p(X)\, \theta'(X, \lambda)\}}{\{\phi(X, \lambda) - \xi p(X)\, \phi'(X, \lambda)\}},$$

where ϕ, θ are the solutions of (1.1) with $\theta(0, \lambda) = 0$, $p(0)\, \theta'(0, \lambda) = 1$, $\phi(0, \lambda) = -1$, $p(\theta)\, \phi'(0, \lambda) = 0$. It is known, see [7], that as X increases the discs $D(X, \lambda)$ nest; and as $X \to \infty$ they converge either to a limit point or a limit disc. We define $m(\lambda)$ to be either the unique limit-point or a fixed point on the boundary of the limit disc. Our concern in this paper is with

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bounds for $|m(\lambda)|$ as $|\lambda| \to \infty$ in sector of the upper half plane. One way of using the above definition of $m(\lambda)$ to obtain such estimates is to estimate a point in $D(X, \lambda)$ for large λ and then to estimate the radius of $D(X, \lambda)$ as $|\lambda| \to \infty$. This gives effective results in the case in which p and w are positive valued functions and is the approach used in [1]. However, in the more general case considered here points within $D(X, \lambda)$ and the radius of $D(X, \lambda)$ are comparable in magnitude. This tends to make the geometric approach to estimating $|m(\lambda)|$ rather cumbersome in this case. In this paper we adopt the more direct approach used by Atkinson in [3].

DEFINITION. $D(X, \lambda)$ consists of those m for which the Ricatti equation

$$v' = -p^{-1} - (\lambda w - q)\, v^2 \tag{1.3}$$

with initial condition

$$v(0) = m \tag{1.4}$$

has a solution over $[0, X]$ which satisfies

$$\operatorname{Im}\{v(X, \lambda)\} \geqslant 0. \tag{1.5}$$

We refer to [3] for a discussion of the equivalence of these, and several other, definitions of $D(X, \lambda)$. For a historical background we refer to the original paper of Weyl [8].

The main result of [3] which we take as our starting point may be stated as follows.

THEOREM A. *For large $|\lambda|$ satisfying (1.2) let $c = c(\lambda)$ be chosen to satisfy*

$$|\lambda|^2 \left(\int_0^c w\, dt\right)\left(\int_0^c w r_1^2\, dt\right) \leqslant 400^{-1} \sin^2 \varepsilon. \tag{1.6}$$

Then any $m \in D(c, \lambda)$ satisfies, for sufficiently large $|\lambda|$,

$$|m| \geqslant \tfrac{1}{2}\, |\lambda| \sin(\varepsilon) \int_0^c w r_1^2\, dt \tag{1.7}$$

$$|1/m| \geqslant \tfrac{1}{2}\, |\lambda| \sin(\varepsilon) \int_0^c w dt, \tag{1.8}$$

where $r_1(t) := \int_0^t p(s)^{-1}\, ds$.

Theorem A is extremely powerful and includes the earlier results of [2] and [6]. It also applies to the particular example of [4] in which $q \equiv 0$ and w, p are powers of x.

It has recently been shown in [5] and the earlier papers referred to therein, that the asymptotic behavior of $m(\lambda)$ is related to the argument of λ. This leads to the question of whether stronger results than Theorem A may be derived if we restrict λ to smaller sectors of the complex plane than that given by (1.2).

2. The Results

For integral N we write

$$S_{N,\varepsilon} = \{\lambda : 0 < \varepsilon < \arg(\lambda) < (2^N - 1)^{-1}(\pi - \varepsilon)\}$$

and $\rho_1(x) := |\int_0^x p(t)^{-1}\, dt|$

$$\rho_j(x) := \int_0^x w(t)\, \rho_{j-1}(t)^2\, dt \qquad \text{for} \quad j = 2, \ldots, N.$$

We suppose throughout that

$$w(t) \geqslant 0 \tag{2.1}$$

and q is such that

$$\left.\begin{aligned} &\int_0^x |q(t)|\, dt = o\left(|\lambda| \int_0^x w\, dt\right) \\ &\int_0^x |q(t)|\, \rho_j(t)\, dt = o\left(|\lambda| \int_0^x w(t)\, \rho_j(t)\, dt\right) \end{aligned}\right\} \tag{2.2}$$

and

$$\int_0^x |q(t)|\, \rho_j(t)^2\, dt = o\left(|\lambda| \int_0^x w(t)\, \rho_j(t)^2\, dt\right) \tag{2.3}$$

as $|\lambda| \to \infty$ for $j = 1, 2$.

We choose $c(\lambda)$ so that

$$|\lambda| \int_0^{c(\lambda)} w(t)\, \rho_1(t)\, dt \leqslant \frac{\delta}{44N} \tag{2.4}$$

$$|\lambda|^2 \int_0^{c(\lambda)} w(t)\, \rho_2(t)\, dt \leqslant \frac{\delta}{44N} \tag{2.5}$$

$$|\lambda|^{2^N} \left(\int_0^{C(\lambda)} w(t)\, dt\right)\left(\int_0^{C(\lambda)} w(t)\, \rho_N(t)^2\, dt\right) \leqslant \frac{\delta}{160}, \tag{2.6}$$

where

$$\delta := 10^{-N-1} \sin(\varepsilon). \tag{2.7}$$

THEOREM 1. *For $\lambda \in S_{N,\varepsilon}$ with $|\lambda|$ sufficiently large, all $m \in D(c(\lambda), \lambda)$ satisfy*

(i) $|m|^{-1} \geqslant \frac{1}{2}\sin(\varepsilon)\, |\lambda| \int_0^{c(\lambda)} w(t)\, dt$

(ii) $|m| \geqslant \frac{1}{2}\sin(\varepsilon)\, |\lambda|^{2^{N-1}} \int_0^{c(\lambda)} w(t)\, \rho_N(t)^2\, dt.$

We remark that the constants appearing in (2.4)–(2.7) are not claimed to be optimal.

3. COMPARISON WITH THEOREM A

In Theorem A Atkinson chooses $c(\lambda)$ so that

$$|\lambda|^2 \left(\int_0^{c(\lambda)} w(t)\, dt\right)\left(\int_0^{c(\lambda)} w(t)\, \rho_1(t)^2\, dt\right) \leqslant 400^{-1} \sin^2 \varepsilon. \tag{3.1}$$

In our notation (3.1) may be rewritten as

$$|\lambda|^2 \left(\int_0^{c(\lambda)} w(t)\, dt\right) \rho_2(c(\lambda)) \leqslant 400^{-1} \sin^2 \varepsilon. \tag{3.2}$$

It follows from the Schwarz inequality and the fact that $\rho_2(\cdot)$ is an increasing function that any $c(\lambda)$ which satisfies (3.2) also satisfies

$$\begin{aligned} |\lambda| \int_0^{c(\lambda)} w(t)\, \rho_1(t)\, dt &\leqslant |\lambda| \left(\int_0^{c(\lambda)} w(t)\, dt\right)^{1/2} \left(\int_0^{c(\lambda)} w(t)\, \rho_1(t)^2\, dt\right)^{1/2} \\ &\leqslant 20^{-1} \sin(\varepsilon) \end{aligned} \tag{3.3}$$

and

$$|\lambda|^2 \int_0^{c(\lambda)} w(t)\, \rho_2(t)\, dt \leqslant |\lambda|^2 \rho_2(c(\lambda)) \int_0^{c(\lambda)} w(t)\, dt \leqslant 400^{-1} \sin^2(\varepsilon). \tag{3.4}$$

The requirements on $c(\lambda)$ given by (2.4) and (2.5) are thus seen to be similar to those of (3.1). It is shown in Lemma 1 that for $m = 2, \ldots, N-1$,

$$\rho_{m+1}(x) \leqslant |\lambda|^{-2^{m-1}} \rho_m(x) \qquad \text{for} \quad 0 \leqslant x \leqslant c(\lambda).$$

Thus, for the $c(\lambda)$ chosen to satisfy (3.1),

$$|\lambda|^{2^N}\left(\int_0^{c(\lambda)} w(t)\,dt\right)\left(\int_0^{c(\lambda)} w(t)\,\rho_N(t)^2\,dt\right) \leqslant |\lambda|^2\left(\int_0^{c(\lambda)} w(t)\,dt\right)\left(\int_0^{c(\lambda)} w(t)\,\rho_1(t)^2\,dt\right) \tag{3.5}$$

for $N \geqslant 2$. The lower bound for $|m|$ given by part (ii) of the Theorem may now be seen to be seen to be at least comparable to the lower bound given by Theorem A.

4. The Power Case

We follow [3, Section 6] and consider, by way of an example, the situation which arises when $q \equiv 0$ and p, w behave, in an average sense, like powers of x. Suppose for $0 \leqslant x \leqslant c(\lambda)$ that

$$c_{11}x^{n(1)} \leqslant \int_0^x w(t)\,dt \leqslant c_{12}x^{n(1)} \tag{4.1}$$

$$\rho_1(x) \leqslant c_{21}x^{n(2)} \tag{4.2}$$

$$\int_0^x w(t)\,\rho_N(t)^2\,dt \geqslant c_{22}x^{n(3)}. \tag{4.3}$$

It follows inductively from (4.1) and (4.2) that there exist a sequence of constants $\{c_j\}$ for $j = 1,..., N$ such that

$$\rho_j(x) \leqslant c_j x^{2^{j-1}(n(1)+n(2))-n(1)} \tag{4.4}$$

for $x \in [0, c(\lambda)]$. Also for $j = 1, 2$ we have that for some constant, C,

$$|\lambda|^{2^{j-1}} \int_0^{c(\lambda)} w(t)\,\rho_j(t)\,dt \leqslant C(|\lambda|\ c(\lambda)^{n(1)+n(2)})^{2^{j-1}} \tag{4.5}$$

and

$$|\lambda|^{2^N}\left(\int_0^{c(\lambda)} w(t)\,dt\right)\left(\int_0^{c(\lambda)} w(t)\,\rho_N(t)^2\,dt\right) \leqslant C(|\lambda|\ c(\lambda)^{n(1)+n(2)})^{2^N}. \tag{4.6}$$

In order to satisfy (2.4)–(2.6) we take, by (4.5) and (4.6),

$$c(\lambda) = K\,|\lambda|^{-1/(n(1)+n(2))} \tag{4.7}$$

for a suitable constant K. Theorem 1 now yields the result that for $\lambda \in S_{N,\varepsilon}$ with $|\lambda|$ sufficiently large

$$|m| \leqslant C\,|\lambda|^{-n(2)/(n(1)+n(2))} \tag{4.8}$$

and

$$|m| \geqslant C\,|\lambda|^{2^N-1-n(3)/(n(1)+n(2))} \tag{4.9}$$

for $m \in D(c(\lambda), \lambda)$.

In the particular case considered in [3] in which (4.3) is replaced by the condition

$$\rho_1(x) \geqslant C_{22}x^{n(2)}$$

we have that $n(3) = 2^N[n(1)+n(2)] - n(1)$ and the bounds of (4.8) and (4.9) agree with those of [3] on the sector $S_{N,\varepsilon}$.

5. Atkinson's Inequality

Our proof of Theorem 1 is based on an inequality derived by Atkinson in [3] which gives necessary conditions on the initial value $v(a)$ in order that the equation

$$v'(x) = -\alpha(x) - \beta(x)\,v(x) - \gamma(x)\,v(x)^2, \qquad 0 \leqslant x \leqslant c \tag{5.1}$$

should have a solution satisfying

$$\operatorname{Im}\{v(c)\} \geqslant 0. \tag{5.2}$$

We suppose that α, β, $\gamma \in L_{\text{loc}}$ and we write

$$\alpha_0 := \int_0^c |\alpha(t)|\,dt \qquad \alpha_1(x) := \int_0^x \alpha(t)\,dt$$

and similarly for β_0, $\beta_1(x)$, γ_0, $\gamma_1(x)$.

Lemma 1. *If* (5.1) *has a solution satisfying* (5.2) *then*

$$|v(0)| \geqslant \operatorname{Im}\{\alpha(c)\} - \alpha_0\{4\beta_0 + 16\alpha_0\gamma_0\},$$
$$|1/v(0)| \geqslant \operatorname{Im}\{\gamma_1(c)\} - \gamma_0\{4\beta_0 + 16\alpha_0\gamma_0\}.$$

This is [3, *Lemma* 1].

6. PRELIMINARY TRANSFORMATIONS

We take as our starting point (5.1). In the case of (1.1) we have, by (1.3), $\alpha := p^{-1}$, $\gamma := \lambda w - q$. Let $v(x, \lambda)$ denote a solution of (5.1) which satisfies (5.2) and let $r(x, \lambda)$ be an absolutely continous function to be chosen below which has

$$\operatorname{Im}\{r(c(\lambda), \lambda)\} \leqslant 0 \qquad \text{for sufficiently large } \lambda \in S_{N,\varepsilon} \tag{6.1}$$

$$r(0, \lambda) = 0. \tag{6.2}$$

It follows from (5.2) and (6.1) that

$$\operatorname{Im}\{v(c(\lambda), \lambda) - r(c(\lambda), \lambda)\} \geqslant 0. \tag{6.3}$$

Moreover, from (5.1), we have

$$(v-r)' = -A - B(v-r) - G(v-r)^2, \tag{6.4}$$

where

$$A := p^{-1} + r' + r^2$$
$$B := 2(\lambda w - q)\, r$$
$$G := \lambda w - q.$$

By (6.30) we may apply Atkinson's inequality, Lemma 1, to (6.4). If, in the notation of Section 5, $4B_0 + 16A_0G_0 < 1$, then for $m \in D(c(\lambda), \lambda)$,

$$|m| = |v(0, \lambda)| \geqslant \operatorname{Im}\{A_1(c(\lambda))\} - A_0\{4B_0 + 16A_0G_0\} \tag{6.5}$$

$$|1/m| = |1/v(0, \lambda)| \geqslant \operatorname{Im}\{G_1(c(\lambda))\} - G_0\{4B_0 + 16A_0G_0\}. \tag{6.6}$$

The object now is to choose r to satisfy (6.1) and (6.2) and to make the right hand sides of (6.5) and (6.6) as large as possible.

We set

$$r(x, \lambda) := \sum_{n=1}^{N} r_n(x, \lambda) \tag{6.7}$$

so that

$$\begin{aligned} A &= p^{-1} + \sum_{n=1}^{N} r_n' + (\lambda w - q) \sum_{n=1}^{N} r_n \sum_{m=1}^{N} r_m \\ &= p^{-1} r_1' + (\lambda w - q)\, r_1^2 + \sum_{n=2}^{N} r_n' \\ &\quad + (\lambda w - q) \sum_{n=2}^{N} r_n \sum_{m=1}^{N} r_m + (\lambda w - q)\, r_1 \sum_{m=2}^{N} r_m. \end{aligned} \tag{6.8}$$

We choose

$$r_1(x, \lambda) := -\int_0^x p(t)^{-1}\, dt$$

and then

$$A = (\lambda w - q)\, r_1^2 + r_2' + 2(\lambda w - q)\, r_1 r_2 + (\lambda w - q)\, r_2^2 \\ + \sum_{n=3}^{N} r_n' + (\lambda w - q) \sum_{n=3}^{N} r_n \sum_{m=1}^{N} r_m + (\lambda w - q) \sum_{n=1}^{2} r_n \sum_{m=3}^{N} r_m.$$

We choose

$$r_2(x, \lambda) := -\int_0^x (\lambda w - q)\, e^{-2 \int_t^x (\lambda w - q)\, r_1\, ds} r_1^2\, dt$$

so that $r_2' + 2(\lambda w - q)\, r_1 r_2 = -(\lambda w - q)\, r_1^2$ and

$$A = (\lambda w - q)\, r_2^2 + \sum_{n=3}^{N} r_n' + (\lambda w - q) \sum_{n=3}^{N} r_n \sum_{m=1}^{N} r_m \\ + (\lambda w - q) \sum_{n=1}^{2} r_n \sum_{m=3}^{N} r_m.$$

We proceed inductively and choose

$$r_{j+1}(x, \lambda) := -\int_0^x (\lambda w - q)\, e^{-2 \int_t^x (\lambda w - q) \sum_{n=1}^{j} r_n(s, \lambda)\, ds} r_j(t, \lambda)^2\, dt \qquad (6.9)$$

for $j = 1, \ldots, N-1$. We then have

$$A(x, \lambda) = (\lambda w - q)\, r_N^2. \qquad (6.10)$$

7. Properties of r

It is clear from (6.9) that the requirement (6.2) is fulfilled. We now show that (6.1) is also satisfied, but first we need some preliminary results.

Lemma 2. *For $j = 3, \ldots, N$,*

$$|\lambda|^{2^{j-1}} \int_0^{c(\lambda)} w(t)\, \rho_j(t)\, dt \leqslant \frac{\delta}{44N}. \qquad (7.1)$$

Proof. We first consider the case $j=3$. Since $\rho_2(\cdot)$ is increasing we have from (2.5) that

$$\rho_3(x)=\int_0^x w(t)\,\rho_2(t)^2\,dt \leqslant \rho_2(x)\int_0^{c(\lambda)} w(t)\,\rho_2(t)\,dt$$

$$\leqslant |\lambda|^{-2}\,\rho_2(x)\,|\lambda|^2\int_0^{c(\lambda)} w(t)\,\rho_2(t)\,dt$$

$$\leqslant \frac{\delta}{44N}\,|\lambda|^{-2}\,\rho_2(x).$$

Thus,

$$|\lambda|^4\int_0^{c(\lambda)} w(t)\,\rho_3(t)\,dt \leqslant \frac{\delta}{44N}\,|\lambda|^2\int_0^{c(\lambda)} w(t)\,\rho_2(t)\,dt \leqslant \frac{\delta}{44N}.$$

Suppose the result holds for ρ_{j-1} then, since $\rho_{j-1}(\cdot)$ is increasing,

$$\rho_j(x)=\int_0^x w(t)\,\rho_{j-1}(t)^2\,dt \leqslant \rho_{j-1}(x)\int_0^{c(\lambda)} w(t)\,\rho_{j-1}(t)\,dt$$

$$\leqslant |\lambda|^{-2^{j-2}}\,\rho_{j-1}(x)\,|\lambda|^{2^{j-2}}\int_0^{c(\lambda)} w(t)\,\rho_{j-1}(t)\,dt$$

$$\leqslant \frac{\delta}{44N}\,|\lambda|^{-2^{j-2}}\,\rho_{j-1}(x).$$

Now,

$$|\lambda|^{2^{j-1}}\int_0^{c(\lambda)} w(t)\,\rho_j(t)\,dt \leqslant |\lambda|^{2^{j-1}-2^{j-2}}\,\frac{\delta}{44N}\int_0^{c(\lambda)} w(t)\,\rho_{j-1}(t)\,dt$$

$$=\frac{\delta}{44N}\,|\lambda|^{-2^{j-2}}\int_0^{c(\lambda)} \omega(t)\,\rho_{j-1}(t)\,dt \leqslant \frac{\delta}{44N}.$$

The proof is now complete by induction.

We note from (2.4), (2.5) that $c(\lambda)\to 0$ as $|\lambda|\to\infty$ and we suppose that $|\lambda|$ is so large that

$$\int_0^x |q(t)|\,dt \leqslant |\lambda|\int_0^x w(t)\,dt \tag{7.2}$$

$$\int_0^x |q(t)|\,\rho_n(t)\,dt \leqslant \frac{\delta}{10}\,|\lambda|\int_0^x w(t)\,\rho_n(t)\,dt \tag{7.3}$$

$$\int_0^x |q(t)|\,\rho_n(t)^2\,dt \leqslant \frac{\delta}{10}\,|\lambda|\int_0^x w(t)\,\rho_n(t)^2\,dt \tag{7.4}$$

for $0\leqslant x\leqslant c(\lambda)$ and $n=1,\ldots,N$.

LEMMA 3. *For $j=2,\dots, N$ and $0 \leqslant x \leqslant c(\lambda)$*

$$|r_j(x, \lambda)+\lambda^{2^{j-1}-1}\rho_j(x)| \leqslant 10^j\delta\, |\lambda|^{2^{j-1}-1}\, \rho_j(x).$$

Proof. We consider first the case $j=2$. We note from (7.2) that

$$\begin{aligned} 2\int_0^{c(\lambda)} |\lambda w-q|\, |r_1|\, dt &\leqslant 2\, |\lambda| \int_0^{c(\lambda)} w\rho_1\, dt+2\int_0^{c(\lambda)} |q|\, \rho_1\, dt \\ &\leqslant \frac{11}{5}\, |\lambda| \int_0^{c(\lambda)} w\rho_1\, dt \leqslant \frac{\delta}{20}. \end{aligned} \tag{7.5}$$

Also,

$$|e^{2\int_0^{c(\lambda)} |\lambda w-q|\, |r_1|\, dt}-1| \leqslant 4\int_0^{c(\lambda)} |\lambda w-q|\, |r_1|\, dt \leqslant \frac{\delta}{10} \tag{7.6}$$

and

$$e^{2\int_0^{c(\lambda)} |\lambda w-q|\, |r_1|\, ds} \leqslant e^{\delta/20} \leqslant \tfrac{5}{4}. \tag{7.7}$$

Now,

$$\begin{aligned} r_2(x, \lambda) &= -\int_0^x (\lambda w-q)\, e^{-2\int_t^x (\lambda w-q)\, r_1\, ds} r_1^2\, dt \\ &= -\lambda\int_0^x wr_1^2\, dt-\lambda\int_0^x w\{e^{-2\int_t^x (\lambda w-q)\, r_1\, ds}-1\}\, r_1^2\, dt \\ &\quad +\int_0^x qe^{-2\int_t^x (\lambda w-q)\, r_1} r_1^2\, dt. \end{aligned}$$

It follows from (7.3)–(7.7) that

$$\begin{aligned} |r_2(x, \lambda)+\lambda\int_0^x wr_1^2\, dt| &\leqslant |\lambda|\, \{e^{2\int_0^{c(\lambda)} |\lambda w-q|\, |r_1|\, ds}-1\} \int_0^x wr_1^2\, dt \\ &\quad + e^{2\int_0^{c(\lambda)} |\lambda w-q|\, |r_1|\, ds} \int_0^x qr_1^2\, dt \\ &\leqslant \delta\cdot 10^2\cdot |\lambda|\, \rho_2(x). \end{aligned}$$

The case $j=2$ is thus established.

Suppose that the result holds for r_n with $n=2,\dots, j$. We now have

$$|r_n(x, \lambda)| \leqslant 2\, |\lambda|^{2^{n-1}-1}\, \rho_n(x) \qquad \text{for} \quad n=1,\dots, j,\ 0 \leqslant x \leqslant c(\lambda) \tag{7.8}$$

and by (7.1)–(7.4),

$$2\int_0^{c(\lambda)} |\lambda w-q| \sum_{n=1}^{j} |r_n|\, ds \leqslant 4\int_0^{c(\lambda)} (|\lambda|\, w+|q|) \sum_{n=1}^{j} |\lambda|^{2^{n-1}} \rho_n\, ds$$

$$\leqslant 4\sum_{n=1}^{j} |\lambda|^{2^{n-1}-1} \left\{|\lambda| \int_0^{c(\lambda)} w\rho_n\, ds+\int_0^{c(\lambda)} |q|\, \rho_n\, ds\right\}$$

$$\leqslant \frac{44}{10} \sum_{n=1}^{j} |\lambda|^{2^{n-1}} \int_0^{c(\lambda)} w\rho_n\, ds$$

$$\leqslant \frac{44}{10} \sum_{n=1}^{j} \frac{\delta}{44N} \leqslant \frac{\delta}{10}. \tag{7.9}$$

Thus,

$$|e^{2\sum_{n=1}^{j} \int_0^{c(\lambda)} |\lambda w-q|\,|r_n|\, ds}-1| \leqslant \frac{\delta}{5} \tag{7.10}$$

and

$$e^{2\sum_{n=1}^{j} \int_0^{c(\lambda)} |\lambda w-q|\,|r_n|\, ds} \leqslant e^{\delta/10} \leqslant \tfrac{5}{4}. \tag{7.11}$$

By the induction hypothesis, (7.8), we have

$$r_j(x, \lambda)= -\lambda^{2^{j-1}-1} \rho_j(x)+\sigma_j(x, \lambda),$$

where $|\sigma_j| \leqslant 10^j \delta\, |\lambda|^{2^{j-1}-1}$. So $r_j(x, \lambda)^2=\lambda^{2^j-2} \rho_j^2-2\lambda^{2^{j-1}-1} \rho_j \sigma_j+\sigma_j^2$ and

$$|r_j(x, \lambda)^2-\lambda^{2^j-2} \rho_j(x)^2| \leqslant 2\cdot\delta\, |\lambda|^{2^j-2}\, 10^j \rho_j^2+\delta^2 10^{2j} \rho_j^2$$

$$\leqslant \delta\cdot 10^j\cdot \rho_j^2 \{2+10^j\cdot\delta\}\, |\lambda|^{2^j-2}$$

$$\leqslant 3\cdot\delta\cdot 10^j \rho_j(x)^2\, |\lambda|^{2^j-2}. \tag{7.12}$$

It follows now from (7.8)–(7.12) that

$$r_{j+1}(x, \lambda)= -\int_0^x (\lambda w-q)\, e^{-2\int_t^x (\lambda w-q) \sum_{n=1}^{j} r_n\, ds} r_j^2\, dt$$

$$= -\lambda^{2^j-1} \int_0^x w\rho_j^2\, dt-\lambda \int_0^x w[r_j^2-\lambda^{2^j-2} \rho_j^2]\, dt$$

$$-\lambda \int_0^x w\{e^{-2\int_t^x (\lambda w-q) \sum_{n=1}^{j} r_n\, ds}-1\}\, r_j^2\, dt$$

$$+\int_0^x q e^{-2\int_t^x (\lambda w-q) \sum_{n=1}^{j} r_n\, ds} r_j^2\, dt.$$

Thus,

$$|r_{j+1}(x,\lambda)+\lambda^{2^j-1}\rho_{j+1}(x)| \leqslant \{3\cdot\delta\cdot 10^j\cdot|\lambda|^{2^j-1}+\frac{\delta}{5}\cdot 2\cdot|\lambda|^{2^j-1}\}\int_0^x w\rho_j^2\,dt$$

$$+\tfrac{5}{4}\cdot 4\cdot|\lambda|^{2^j-2}\int_0^x |q|\,\rho_j^2\,dt$$

$$\leqslant \delta\cdot\{3\cdot 10^j+\tfrac{2}{5}+\tfrac{5}{100}\}\,|\lambda|^{2^j-1}\int_0^x w\rho_j^2\,dt$$

$$\leqslant \delta\cdot 10^j\cdot|\lambda|^{2^j-1}\int_0^x w\rho_j^2\,dt.$$

The proof is now complete.

We consider now the requirement (6.1). Suppose $\lambda =: |\lambda|\,e^{i\theta}$; then for $|\lambda|$ sufficiently large,

$$\begin{aligned}\operatorname{Im}\{r_j(c(\lambda),\lambda)\} &\leqslant -\operatorname{Im}\{\lambda^{2^{j-1}-1}\rho_j(c(\lambda))\}+10^j\cdot\delta\cdot|\lambda|^{2^{j-1}-1}\rho_j(c(\lambda))\\ &\leqslant -|\lambda|^{2^{j-1}-1}\rho_j(c(\lambda))\{\sin[(2^{j-1}-1)\,\theta]-10^j\delta\}\\ &\leqslant 0\end{aligned}$$

since $\delta := 10^{-N-1}\sin(\varepsilon)$ and $0<\varepsilon<\theta<(2^N-1)^{-1}(\pi-\varepsilon)$.

8. Proof of Theorem 1

We consider first the quantity $4B_0+16A_0G_0$. By (6.10)

$$4B_0+16A_0G_0 = 8\int_0^{c(\lambda)} |\lambda w-q|\,|r|\,dt+16\left(\int_0^{c(\lambda)} |p^{-1}+r'+(\lambda w-q)\,r^2|\,dt\right)$$

$$\times\left(\int_0^{c(\lambda)} |\lambda w-q|\,dt\right)$$

$$\leqslant 8\sum_{n=1}^{N}\int_0^{c(\lambda)} |\lambda w-q|\,|r_n|\,dt+16\left(\int_0^{c(\lambda)} |\lambda w-q|\,dt\right)$$

$$\times\left(\int_0^{c(\lambda)} |\lambda w-q|\,|r_N|^2\,dt\right)$$

$$\leqslant 16 \sum_{n=1}^{N} \int_0^{c(\lambda)} |\lambda w - q|\, |\lambda|^{2^{n-1}-1} \rho_n\, dt$$

$$+ 48 \left(\int_0^{c(\lambda)} |\lambda w - q|\, dt \right) \left(\int_0^{c(\lambda)} |\lambda w - q|\, |\lambda|^{2^N - 2} \rho_N^2\, dt \right)$$

$$\leqslant 16 \cdot \frac{11}{10} \sum_{n=1}^{N} |\lambda|^{2^{n-1}} \int_0^{c(\lambda)} w\rho_n\, dt + 48 \cdot \left(\frac{11}{10}\right)^2 |\lambda|^{2^N}$$

$$\times \left(\int_0^{c(\lambda)} w dt \right) \left(\int_0^{c(\lambda)} w\rho_N^2\, dt \right)$$

$$\leqslant \frac{16 \cdot 11}{10} \sum_{n=1}^{N} \frac{\delta}{44N} + \frac{48 \cdot 11^2 \cdot \delta}{10^2 \cdot 160} \leqslant \delta \tag{8.1}$$

by (2.4)–(2.6) and Lemma 2.

We first consider the bound (i). By (6.5) we have that for all $m \in D(c(\lambda), \lambda)$ and $|\lambda|$ sufficiently large,

$$|m|^{-1} \geqslant \operatorname{Im} \left\{ \int_0^{c(\lambda)} |\lambda w - q)\, dt \right\} - \delta \int_0^{c(\lambda)} |\lambda w - q|\, dt$$

$$\geqslant \operatorname{Im}\{\lambda\} \int_0^{c(\lambda)} w\, dt - \delta\, |\lambda| \int_0^{c(\lambda)} w\, dt - \frac{2\delta}{10} |\lambda| \int_0^{c(\lambda)} w\, dt$$

$$\geqslant |\lambda| \sin(\varepsilon) \left(\int_0^{c(\lambda)} w\, dt \right) \{1 - 10^{-N-1} - 2 \cdot 10^{-N-2}\}$$

$$\geqslant \tfrac{1}{2} |\lambda| \sin(\varepsilon) \left(\int_0^{c(\lambda)} w\, dt \right) \text{ by (7.2).}$$

We now look at (ii). By (6.5) we have that for all $m \in D(c(\lambda), \lambda)$ and $|\lambda|$ sufficiently large

$$|m| \geqslant \operatorname{Im} \left\{ \int_0^{c(\lambda)} (\lambda w - q)\, r_N^2\, dt \right\} - \delta \int_0^{c(\lambda)} |\lambda w - q|\, |r_N|^2\, dt$$

$$\geqslant \operatorname{Im} \left\{ \lambda \left\{ \int_0^{c(\lambda)} w r_N^2\, dt \right\} - \operatorname{Im} \left\{ \int_0^{c(\lambda)} q r_N^2\, dt \right\} \right.$$

$$- \delta \int_0^{c(\lambda)} |\lambda w|\, |r_N|^2\, dt - \delta \int_0^{c(\lambda)} |q|\, |r_N|^2\, dt$$

$$= \operatorname{Im}\{\lambda^{2^N - 1}\} \int_0^{c(\lambda)} w\rho_N^2\, dt + \operatorname{Im} \left\{ \lambda \int_0^{c(\lambda)} w[r_N^2 - \lambda^{2^N - 2} \rho_N^2]\, dt \right\}$$

$$- \operatorname{Im} \left\{ \int_0^{c(\lambda)} q r_N^2\, dt \right\} - \delta \int_0^{c(\lambda)} |\lambda w|\, |r_N|^2\, dt - \delta \int_0^{c(\lambda)} |q|\, |r_N|^2\, dt$$

$$\geqslant \Big\{ |\lambda|^{2^N-1}\sin(\varepsilon) - 3\cdot\delta\cdot 10^N\,|\lambda|^{2^N-1} - \frac{3\delta}{100}\,|\lambda|^{2^{N-1}}$$

$$3\cdot\delta\cdot|\lambda|^{2^N-1} - \frac{3\delta^2}{100}\,|\lambda|^{2^{N-1}}\Big\}\int_0^{c(\lambda)} w\rho_N^2\,dt$$

$$\geqslant |\lambda|^{2^N-1}\left(\int_0^{c(\lambda)} w\rho_N^2\,dt\right)\sin(\varepsilon)\{1-\delta\cdot\operatorname{cosec}(\varepsilon)[3\cdot 10^N + \tfrac{3}{100} + 3 + 3]\}$$

$$\geqslant |\lambda|^{2^N-1}\sin(\varepsilon)\left(\int_0^{c(\lambda)} w\rho_N^2\,dt\right)\{1-\delta\cdot 4\cdot 10^N\operatorname{cosec}(\varepsilon)\}.$$

$$\geqslant \tfrac{1}{2}\,|\lambda|^{2^N-1}\sin(\varepsilon)\int_0^{c(\lambda)} w\rho_N^2\,dt$$

Since $\delta = 10^{-N-1}\sin(\varepsilon)$. The result now follows.

References

1. F. V. Atkinson, On the location of the Weyl circles, *Proc. Roy. Soc. Edinburgh Sect. A* **88** (1981), 345–356.
2. F. V. Atkinson, On bounds for the Titchmarsh–Weyl *m*-coefficients and for spectral functions for second order differential operations, *Proc. Roy. Soc. Edinburgh Sect. A* **97** (1984), 1–7.
3. F. V. Atkinson, On the order of magnitude of Titchmarsh–Weyl functions, *Differential Integral Equations* **1** (1988), 79–96.
4. W. N. Everitt and A. Zettl, On a class of integral inequalities, *J. London Math. Soc. (2)* **17** (1978), 291–303.
5. B. J. Harris, An inverse problem involving the Titchmarsh–Weyl *m*-function, *Proc. Roy. Soc. Edinburgh Sect. A* **110** (1988), 305–309.
6. E. Hille, Green's transforms, and singular boundary value problems, *J. Math. Pures Appl. (9)* **42** (1963), 331–349.
7. E. C. Titchmarsh, "Eigenfunction Expansion Associated with Second Order Differential Equations," Vol. 1, 2nd ed., Oxford Univ. Press, London/New York, 1962.
8. H. Weyl, Ueber gewöhnliche Differentialgleichen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Functionen, *Math. Ann.* **68** (1910), 220–269.